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ORIGINAL PAPER

A simple proof of the continuity of expected payoffs

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Abstract

In n-person games with compact strategy spaces and continuous payoff functions, the expected payoff functions of the mixed extension are continuous with respect to the product of the weak* topologies. We provide an elementary proof of this fact. The analysis reveals that the Hausdorff separation axiom, commonly imposed on the topology of players' strategy spaces, is not needed. We also show that the definition of expected payoffs as an iterated integral does not depend on the ordering of the players.

Keywords Compact games · Expected payoffs · Weak* topology · Continuity · Hausdorff separation axiom

JEL Classification C72

1 Introduction

For noncooperative n-player games with compact Hausdorff strategy spaces and continuous payoff functions, Glicksberg (1952) observed that a player's expected payoff is continuous with respect to the product of the weak* topologies on the corresponding spaces of mixed strategies. As the original work did not provide any further details, a small but significant literature has since developed, offering complete proofs of this observation under varying sets of assumptions and employing a range

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of techniques. Specifically, Glycopantis and Muir (2000) applied the Stone–Weierstrass Theorem, while Zarichnyi (2004) relied on Milyutin maps. Focusing on the case of metrizable strategy spaces, Aliprantis et al. (2006) used tools from functional analysis, such as the Closed Graph Theorem.[1].

In this note, we offer an alternative proof of the continuity of expected payoffs that avoids the use of advanced techniques, relying instead on an elementary characterization of continuity of a real-valued function on a product of two topological spaces. The analysis shows, in particular, that the Hausdorff separation axiom, which is usually imposed on players' strategy spaces, can be dropped without losing the conclusion of the theorem. We exploit the same methods to explain why the iterated-integrals definition of a player's expected payoff does not depend on the order of integration.

The characterization of continuity of an arbitrary mapping on a product space is, in general, difficult. Munkres (2014, Ch. 2, p. 110) wrote that there is "no useful criterion" for the continuity of a mapping that is defined on the product of topological spaces. Because payoff functions are real-valued, however, the situation changes. Recall that a real-valued function $f(x, y)$ defined on a product of topological spaces is called *separately continuous* in x if for any given (x_0, y_0) and every $\varepsilon > 0$, we can find an open neighborhood N of x_0 such that $|f(x, y_0) - f(x_0, y_0)| < \varepsilon$ for all $x \in N$. Separate continuity in y is defined analogously. Then, clearly, any jointly continuous $f(x, y)$ is separately continuous both in x and in y. For compact strategy spaces, this implication can be reversed if the continuity requirement with respect to one of the variables is strengthened as follows. We will say that f is continuous in x, *uniformly* over y, if for any x_0 and $\varepsilon > 0$, we can find an open neighborhood N of x_0 such that $|f(x, y) - f(x_0, y)| < \varepsilon$ for all $x \in N$ and all y. Then, our characterization says that $f(x, y)$ is jointly continuous if and only if $f(x, y)$ is continuous in x, uniformly over y, as well as separately continuous in y. Since continuity in x, uniformly over y, is preserved under taking expectations with respect to y, this characterization provides exactly the right tool to "lift" continuity properties from a game to its mixed extension.

The remainder of this note is organized as follows. Section 2 presents our characterization of joint continuity. Section 3 applies this result to establish the continuity of expected payoffs in the mixed extension. In Sect. 4, we prove a complementary result concerning the order of integration.

2 A characterization of joint continuity

The proof of our main result is based on the following characterization of the continuity of a function defined on the product of two topological spaces.

Lemma 1 *Let X and Y be topological spaces, and let $f : X \times Y \to \mathbb{R}$. If f is continuous and Y is compact, then:*

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[1] For related results, see also Becker and Damianov (2006), Kozhan and Zarichnyi (2008), and Kim (2014).

(i) *f is continuous in x, uniformly over $y \in Y$; and*
(ii) *f is continuous in y, for every $x \in X$.Conversely, if conditions (i) and (ii) hold, then f is continuous on $X \times Y$ (even if Y is not compact).*

Proof [2] Fix $x_0 \in X$ and $\varepsilon > 0$. For each $y \in Y$, by the continuity of f at (x_0, y) there exist open neighborhoods $N(y)$ of x_0 and $V(y)$ of y such that $|f(x, y') - f(x_0, y)| < \frac{\varepsilon}{2}$, for all $x \in N(y)$ and $y' \in V(y)$. Taking $x = x_0$ yields $|f(x_0, y') - f(x_0, y)| < \frac{\varepsilon}{2}$ for all $y' \in V(y)$. Hence, $|f(x, y') - f(x_0, y')| < \varepsilon$ for all $x \in N(y)$ and $y' \in V(y)$. The family $\{V(y)\}_{y \in Y}$ covers Y, and by compactness there exist $y_1, \ldots, y_K \in Y$ such that $Y = \bigcup_{k=1}^{K} V(y_k)$. Let $N = \bigcap_{k=1}^{K} N(y_k)$. Then N is an open neighborhood of x_0 and $|f(x, y) - f(x_0, y)| < \varepsilon$, for all $x \in N$ and $y \in Y$, which proves (i). Condition (ii) is immediate by fixing x.

Conversely, fix $(x_0, y_0) \in X \times Y$ and $\varepsilon > 0$. By (i), there exists an open neighborhood N of x_0 such that $|f(x, y) - f(x_0, y)| < \frac{\varepsilon}{2}$ for all $x \in N$ and all $y \in Y$. By (ii), there exists an open neighborhood VV of y_0 such that $|f(x_0, y) - f(x_0, y_0)| < \frac{\varepsilon}{2}$ for all $y \in V$. Thus, $|f(x, y) - f(x_0, y_0)| < \varepsilon$, for all $(x, y) \in N \times V$. Hence$f$is continuous at (x_0, y_0), and therefore jointly continuous on $X \times Y$.[3] $\square$

3 The continuity of expected payoffs

Let $G = (S_i, u_i)_{i=1}^{n}$ be an n-person noncooperative game in strategic form, where S_i is the set of player i's pure strategies and

$$u_i : S_1 \times \ldots \times S_n \to \mathbb{R}$$

is player i's payoff function, for $i \in \{1, \ldots, n\}$. It is assumed that each S_i is compact, but not necessarily Hausdorff, and that each u_i is jointly continuous. Let $\mathcal{P}(S_i)$ denote the space of all regular[4] probability measures on the Borel subsets of S_i, with typical element μ_i. In the *mixed extension* of G, each player $i \in \{1, \ldots, n\}$ chooses

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[2] The first part of Lemma 1 corresponds to Cohn (2013, Lem. 7.6.3). The proof of that part is, consequently, added for completeness only.

[3] As kindly pointed out by the Associate Editor, the lemma can alternatively be shown using nets. We illustrate this for the second part of the proof. Let (x_α, y_α) be a net converging to $(x_0, y_0) \in X \times Y$ and let $\varepsilon > 0$. By (i), there exists α_1 such that $|f(x_\alpha, y) - f(x_0, y)| < \frac{\varepsilon}{2}$, for any $\alpha \geq \alpha_1$ and $y \in Y$. In particular, $|f(x_\alpha, y_\alpha) - f(x_0, y_\alpha)| < \frac{\varepsilon}{2}$, for any $\alpha \geq \alpha_1$. By (ii), there exists α_2 such that $|f(x_0, y_\alpha) - f(x_0, y_0)| < \frac{\varepsilon}{2}$ for any $\alpha \geq \alpha_2$. Hence, $|f(x_\alpha, y_\alpha) - f(x_0, y_0)| < \varepsilon$, for any $\alpha \geq \max\{\alpha_1, \alpha_2\}$.

[4] Following Dunford and Schwartz (1958, Sec. III.5), we call a probability measure μ defined on the Borel sets of a topological space X *regular* if for any Borel set B and $\varepsilon > 0$, there is a closed set F and an open set U such that $F \subseteq B \subseteq U$ and $\mu(U \backslash F) < \varepsilon$.

some $\mu_i \in \mathcal{P}(S_i)$.[5] Glicksberg (1952) defined player i's expected payoff as the iterated expectation

$$E_{\mu_1,\ldots,\mu_n}[u_i] = \int_{S_1} \left\{ \cdots \left\{ \int_{S_n} u_i(s_1,\ldots,s_n) d\mu_n(s_n) \right\} \cdots \right\} d\mu_1(s_1).$$

We may use Lemma 1 to check that the iterated integral is well-defined. From continuity, $u_i(s_1,\ldots,s_n)$ is continuous in $(s_1,\ldots,s_{n-1})$, uniformly over s_n. Hence, the innermost integral $\int_{S_n} u_i(s_1,\ldots,s_n) d\mu_n(s_n)$ is continuous in $(s_1,\ldots,s_{n-1})$. Straightforward induction over n yields the claim.[6]

Let $\mathcal{C}(S_i)$ denote the space of continuous (and therefore bounded) functions on S_i. We define the *weak* topology* on $\mathcal{P}(S_i)$ as the coarsest topology such that the evaluation map $e_f : \mu_i \mapsto \int_{S_i} f(s_i)\, d\mu_i(s_i)$ is continuous, for any $f \in \mathcal{C}(S_i)$. Glicksberg (1952) introduced the weak* topology by embedding $\mathcal{P}(S_i)$ into the adjoint of the Banach space $\mathcal{C}(S_i)$, i.e., via the Riesz Representation Theorem. In the Hausdorff case, our definition of the weak* topology coincides with Glicksberg's definition.[7]

The following result establishes the continuity of expected payoffs in games with compact pure strategy spaces.

Theorem 1 *Suppose that $S_1,\ldots,S_n$ are compact (but not necessarily Hausdorff), and that u_i is continuous. Then, the mapping $(\mu_1,\ldots,\mu_n) \mapsto E_{\mu_1,\ldots,\mu_n}[u_i]$ is continuous with respect to the product of the weak* topologies.*

Proof By Lemma 1, $u_i(s_1,\ldots,s_n)$ is continuous in $(s_1,\ldots,s_{n-1})$, uniformly over s_n. Fix $(s_1^0,\ldots,s_{n-1}^0) \in S_1 \times \cdots \times S_{n-1}$ and $\varepsilon > 0$. Then, there exists a neighborhood N of $(s_1^0,\ldots,s_{n-1}^0)$ such that $|u_i(s_1,\ldots,s_{n-1},s_n) - u_i(s_1^0,\ldots,s_{n-1}^0,s_n)| < \varepsilon$ for all $(s_1,\ldots,s_{n-1}) \in N$ and all $s_n \in S_n$. Hence for every $\mu_n \in \mathcal{P}(S_n)$,

$$\begin{aligned}
&\left| \int_{S_n} u_i(s_1,\ldots,s_{n-1},s_n)\, d\mu_n(s_n) - \int_{S_n} u_i(s_1^0,\ldots,s_{n-1}^0,s_n)\, d\mu_n(s_n) \right| \\
&\le \int_{S_n} \left| u_i(s_1,\ldots,s_{n-1},s_n) - u_i(s_1^0,\ldots,s_{n-1}^0,s_n) \right| d\mu_n(s_n) \\
&< \varepsilon.
\end{aligned}$$

Thus,

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[5] In the Hausdorff case, this is the standard definition. For the general non-Hausdorff case, our definition is identical to that used by Mertens (1986, p. 243). To ensure that $\mathcal{P}(S_i)$ does not exclude pure strategies, one might want to impose that the topology on S_i satisfies the T_1 separation axiom (cf. Ewerhart (2025b)). However, that assumption is not needed in the sequel.

[6] The reader might wonder if the value of the iterated integral is independent of the order of integration. This is indeed the case, as we explain in Sect. 4.

[7] Indeed, the weak* topology on the subset is trivially identical to the corresponding subspace topology (e.g., Bourbaki, 1995, §2.3, Prop. 5).

$$E_{\mu_n}[u_i] = \int_{S_n} u_i(s_1, \ldots, s_n)\, d\mu_n(s_n)$$

is continuous in $(s_1, \ldots, s_{n-1})$, uniformly over μ_n. Further, by the definition of the weak* topology, $E_{\mu_n}[u_i]$ is continuous in μ_n, for any fixed $(s_1, \ldots, s_{n-1})$. Therefore, using Lemma 1 again, $E_{\mu_n}[u_i]$ is continuous in $(s_1, \ldots, s_{n-1}, \mu_n)$, hence continuous in $(s_1, \ldots, s_{n-2}, \mu_n)$, uniformly over s_{n-1}. By the argument detailed above, $E_{\mu_{n-1},\mu_n}[u_i] = E_{\mu_{n-1}}[E_{\mu_n}[u_i]]$ is continuous in $(s_1, \ldots, s_{n-2}, \mu_{n-1}, \mu_n)$. Proceeding by induction, $E_{\mu_1,\ldots,\mu_n}[u_i] = E_{\mu_1}[\ldots[E_{\mu_n}[u_i]]\ldots]$ is seen to be continuous in $(\mu_1, \ldots, \mu_n)$, as has been claimed. □

4 The order of integration

In this section, we return to the issue raised in Footnote [8]. Specifically, we show that the iterated-integral definition of expected payoffs does not depend on the order of integration. While this result is not needed to establish the continuity of expected payoffs, it is nevertheless important from an applied perspective because it is usually understood that players can be renamed without changing the nature of the game. For convenience, we derive the result in a slightly more general form for (not necessarily regular) Borel probability measures. But clearly, it holds also if players are constrained to choose regular strategies, i.e., in the mixed extension.

Given a topological space X, we denote by $\widehat{\mathcal{P}}(X) \supseteq \mathcal{P}(X)$ the set of Borel probability measures. We will prove the following result.

Theorem 2 *Let $S_1, \ldots, S_n$ be compact topological spaces. For any Borel probability measures $\mu_1 \in \widehat{\mathcal{P}}(S_1), \ldots, \mu_n \in \widehat{\mathcal{P}}(S_n)$ and any continuous function u_i on $S_1 \times \ldots \times S_n$, we have*

$$E_{\mu_1,\ldots,\mu_n}[u_i] = E_{\mu_{\tau(1)},\ldots,\mu_{\tau(n)}}[u_i],$$

for any permutation $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$ of the player set, where the expectation on the right-hand side is defined as

$$E_{\mu_{\tau(1)},\ldots,\mu_{\tau(n)}}[u_i] = \int_{S_{\tau(1)}} \left\{ \ldots \left\{ \int_{S_{\tau(n)}} u_i(s_1, \ldots, s_n) d\mu_{\tau(n)}(s_{\tau(n)}) \right\} \ldots \right\} d\mu_{\tau(1)}(s_{\tau(1)}).$$

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[8] 6

Proof [9] We recall first that any permutation of the set of players can be decomposed into a finite sequence of swaps (i.e., transpositions) between two consecutive players j and $j+1$, where $j \in \{1, \ldots, n-1\}$. It therefore suffices to show that

$$E_{\mu_1,\ldots,\mu_n}[u_i] = E_{\mu_1,\ldots,\mu_{j-1},\mu_{j+1},\mu_j,\mu_{j+2},\ldots,\mu_n}[u_i],$$

for any such j. For this, fix $(s_1, \ldots, s_{j-1}) \in S_1 \times \ldots \times S_{j-1}$ as well as $(\mu_{j+1}, \ldots, \mu_n) \in \widehat{\mathcal{P}}(S_{j+1}) \times \ldots \times \widehat{\mathcal{P}}(S_n)$. Consider the difference

$$\Delta(\mu_j) = E_{\mu_j,\ldots,\mu_n}[u_i] - E_{\mu_{j+1},\mu_j,\mu_{j+2},\ldots,\mu_n}[u_i],$$

where

$$E_{\mu_j,\ldots,\mu_n}[u_i] = \int_{S_j} \left\{ \ldots \left\{ \int_{S_n} u_i(s_1, \ldots, s_n) d\mu_n(s_n) \right\} \ldots \right\} d\mu_j(s_j)$$

and

$$\begin{aligned} &E_{\mu_{j+1},\mu_j,\mu_{j+2},\ldots,\mu_n}[u_i] \\ &= \int_{S_{j+1}} \int_{S_j} \int_{S_{j+2}} \left\{ \ldots \left\{ \int_{S_n} u_i(s_1, \ldots, s_n) d\mu_n(s_n) \right\} \ldots \right\} \\ &d\mu_{j+2}(s_{j+2}) d\mu_j(s_j) d\mu_{j+1}(s_{j+1}). \end{aligned}$$

Further, note that the proof of Theorem 1 does not make use of the regularity assumption. Therefore, applying the corresponding variant of the theorem to the payoff function $u_i(s_1, \ldots, s_{j-1}, \cdot)$ between the players $j, \ldots, n$, it follows that the function $\Delta(\mu_j)$ is weak* continuous on $\widehat{\mathcal{P}}(S_j)$. Moreover, by the linearity of the Lebesgue integral, $\Delta(\mu_j)$ vanishes on the set of convex combinations of Dirac measures on S_j. By Lemma 2 below, the set of convex combinations of Dirac measures on S_j is weak* dense in $\widehat{\mathcal{P}}(S_j)$. It follows that

$$E_{\mu_j,\ldots,\mu_n}[u_i] = E_{\mu_{j+1},\mu_j,\mu_{j+2},\ldots,\mu_n}[u_i],$$

for any $(s_1, \ldots, s_{j-1}) \in S_1 \times \ldots \times S_{j-1}$ and $(\mu_j, \ldots, \mu_n) \in \widehat{\mathcal{P}}(S_j) \times \ldots \times \widehat{\mathcal{P}}(S_n)$. Taking the expectation $E_{\mu_1,\ldots,\mu_{j-1}}[\cdot]$ on both sides shows that swapping the order of integration between players j and $j+1$ indeed does not affect the value of the iterated integral. The claim follows. □

The following lemma is used in the proof of Theorem 2.

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[9] The proof is inspired by ideas contained in Glicksberg (1962, Theorem. 3.1) and Bogachev (2007, Ex. 8.1.6, p. 176). An alternative strategy of proof for Theorem 2 entails first showing that u_i is measurable with respect to the product of the respective Borel σ-algebras on $S_1, \ldots, S_n$, and then applying Fubini's theorem. That approach is no longer elementary but leads, of course, to the additional conclusion that the iterated expectation coincides with the joint expectation with respect to the product measure. For details, see Ewerhart (2025a).

Lemma 2 *Let X be a topological space. Then, the set of convex combinations of Dirac measures on X is weak* dense in $\widehat{\mathcal{P}}(X)$.*

Proof For a topological space X, a base of the weak* topology on $\widehat{\mathcal{P}}(X)$ is the collection of all sets

$$\begin{aligned} & U_{f_1,\ldots,f_K,\varepsilon}(\mu) \\ & = \left\{ \nu \in \widehat{\mathcal{P}}(X) : \left| \int_X f_k(x)d\nu(x) - \int_X f_k(x)d\mu(x) \right| < \varepsilon, k = 1,\ldots,K \right\}, \end{aligned}$$

where μ is a Borel probability measure on X, $f_1,\ldots,f_K$ are continuous and bounded functions on X, and $\varepsilon > 0$. It suffices to show that any such set contains a convex combination of Dirac measures. For this, choose, for any k, a simple function g_k such that $|f_k(x) - g_k(x)| < \frac{\varepsilon}{2}$ for any $x \in X$. Select an element from each Borel set $B \neq \varnothing$ in the join of the partitions associated with $g_1,\ldots,g_K$, and attach the weight $\mu(B)$ to the corresponding Dirac measure. This delivers a probability measure ν^* that is a convex combination of Dirac measures. Moreover, by construction, $\int_X g_k(x)d\mu(x) = \int_X g_k(x)d\nu^*(x)$, so that by a straightforward application of the triangle inequality,

$$\begin{aligned} & \left| \int_X f_k(x)d\nu^*(x) - \int_X f_k(x)d\mu(x) \right| \\ & \leq \left| \int_X f_k(x)d\nu^*(x) - \int_X g_k(x)d\nu^*(x) \right| + \left| \int_X g_k(x)d\mu(x) - \int_X f_k(x)d\mu(x) \right| \\ & \leq \int_X |f_k(x) - g_k(x)|\, d\nu^*(x) + \int_X |g_k(x) - f_k(x)|\, d\mu(x) < \varepsilon, \end{aligned}$$

for all $k \in \{1,\ldots,K\}$, i.e., ν^* is contained in $U_{f_1,\ldots,f_K,\varepsilon}(\mu)$. □

Declarations

Conflict of interest There are no conflicts of interest to declare.

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